JOURNAL OF COMPUTATIONAL PHYSICS 29, 421–430 (1978)

Numerical Testing of Minimum-Delay, Positive-Real, and Positive-Definite Digital Filters

ENDERS A. ROBINSON,* DAN LOEWENTHAL,† AND SVEN TREITEL*

*Research Center, Amoco Production Company, Tulsa, Oklahoma, and
† Department of Geophysics and Planetary Sciences, Tel Aviv University, Ramat Aviv, Israel

Received April 12, 1978

Various types of digital filters are characterized by their phase spectra. The phase spectrum of a symmetric filter can only take on values that are integral multiples of π. The phase spectrum of a positive-definite filter is zero for all frequencies. The phase spectrum of a minimum-delay filter must have the same value at the negative Nyquist frequency as at the positive Nyquist frequency, so that its net phase change over the Nyquist frequency range is zero. A positive-real filter in addition to having a zero net phase change over the Nyquist frequency range must have a phase spectrum that is less than $\pi/2$ in magnitude. A reflection coefficient theorem is established which states that a filter is minimum-delay if and only if its associated reflection coefficients are less than one in magnitude. A positive-definite theorem is established which states that a filter is positive-definite if and only if a Chebyshev-related polynomial has no real roots in magnitude less than or equal to one. Numerical tests for minimum-delay, positive-definite, and positive-real are given based on these two theorems.

1. POSITIVE-DEFINITE FILTERS

The term "digital filter" is commonly used to mean a linear time-invariant filter in discrete time. Time is represented by the integer n, and signals x_n are represented by sequences indexed by the integer n. In this paper we consider only real-valued signals, but our results can readily be extended to the complex-valued case. The output y_n of a digital filter is given as the convolution of the input x_n with the filter weights c_n ; that is,

$$y_n = \sum_{k=-\infty}^{\infty} c_k x_{n-k} .$$

For stability, a condition on the filter weights such as

$$\sum_{n=-\infty}^{\infty} c_n^2 < \infty$$

is imposed. The z-transform of the digital filter is defined as the formal power series

$$C(z) = \sum_{n=-\infty}^{\infty} c_n z^n.$$

0021-9991/78/0293–0421$02.00/0

581/29/3-8

The frequency spectrum of the digital filter can be defined as

$$C(e^{i\omega}) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega n}.$$

The real variable ω is angular frequency[1] which is expressed in radians per discrete time unit. Because $C(e^{i\omega})$ is periodic with period 2π, we can consider ω only in the so-called Nyquist range $-\pi \leqslant \omega \leqslant \pi$. Here $\omega = \pi$ is the Nyquist frequency and $\omega = -\pi$ is the negative Nyquist frequency. The filter weights are uniquely determined from the frequency spectrum by the expression

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} C(e^{i\omega})\, e^{-i\omega n}\, d\omega. \tag{1}$$

For real-valued weights (which we are considering in this paper), the real and imaginary parts of the frequency spectrum are, respectively, the cosine transform and the sine transform of the weights; that is,

$$\operatorname{Re} C(e^{i\omega}) = \sum_{n=-\infty}^{\infty} c_n \cos \omega n,$$

$$\operatorname{Im} C(e^{i\omega}) = \sum_{n=-\infty}^{\infty} c_n \sin \omega n.$$

For real-valued weights, the frequency spectrum has the properties: (1) its magnitude $| C(e^{i\omega})|$, called the magnitude spectrum, is an even function of ω, i.e., $| C(e^{i\omega})| = | C(e^{-i\omega})|$, and (2) its argument $\arg C(e^{i\omega})$, called the phase spectrum $\Theta(\omega) = \arg C(e^{i\omega})$, is an odd function, that is, $\Theta(\omega) = -\Theta(-\omega)$.

If the weights are symmetric, then the sine transform is zero. That is, if $c_{-n} = c_n$, then

$$\sum_{n=-\infty}^{\infty} c_n \sin \omega n = 0.$$

Such filters are called *symmetric filters*. The frequency spectrum $C(e^{i\omega})$ of a symmetric filter is real-valued, and hence its phase spectrum can only assume values which are integral multiples of π. More specifically, the phase spectrum $\Theta(\omega)$ of a symmetric filter can take on only the values $0, \pm 2\pi, \pm 4\pi, \pm 6\pi$ in those regions of ω where $C(e^{i\omega}) > 0$ and $\pm\pi, \pm 3\pi, \pm 5\pi, ...$ in those regions of ω where $C(e^{i\omega}) < 0$. A *zero-phase-shift filter* is defined as a filter for which $\Theta(\omega) = 0$. Clearly, a filter is zero-phase shift if and only if $C(e^{i\omega})$ is realvalued and positive for all ω. Symmetric filters have real-valued but not necessarily positive frequency functions, so the zero-phase-shift filters make up a subclass of the class of all symmetric filters.

[1] Our angular frequency variable ω is the negative of the angular frequency variable commonly used by engineers.

A digital filter with (real-valued) weights c_n is called *positive-definite* if for every positive integer M and every set of real numbers $x_0, x_1, \ldots, x_M$ the quadratic form

$$Q = \sum_{j=0}^{M} \sum_{k=0}^{M} x_j c_{k-j} x_k$$

is positive. A quadratic form such as this one in which the kernel appears as c_{k-j} rather than as $c_{j,k}$ is called a *Toeplitz form*. Making use of Eq. (1), the quadratic form may be written as

$$Q = \frac{1}{2\pi} \sum_{j=0}^{M} \sum_{k=0}^{M} x_j x_k \int_{-\pi} C(e^{i\omega})\, e^{-i\omega(k-j)}\, d\omega,$$

which is

$$Q = \frac{1}{2\pi} \int_{-\pi}^{\pi} C(e^{i\omega}) \left| \sum_{j=0}^{M} x_j e^{i\omega j} \right|^2 d\omega.$$

Since M and the $x_0, x_1, \ldots, x_M$ are completely arbitrary, we see that Q and $C(e^{i\omega})$ must be positive together. Thus a positive frequency spectrum yields a positive-definite filter and conversely a positive-definite filter has a positive frequency spectrum. Thus, in order for a filter to be zero-phase-shift it is necessary and sufficient for the filter to be positive-definite. Note that we consider any phase that is an integer multiple of 2π as effectively being the same as zero phase. In summary, a symmetric filter has a real frequency spectrum, whereas a positive-definite filter has a positive frequency spectrum.

2. Minimum-Delay Filters

Symmetric filters are necessarily two-sided, that is, c_n can be nonzero for negative as well as positive values of n. As we have seen in the foregoing section, symmetric filters are characterized by their phase spectra. A symmetric filter is a filter with a phase spectrum that can only take on values that are integral multiples of π. A subclass of the symmetric filters are the positive-definite filters. A positive-definite filter is a filter with a phase spectrum that can only take on values that are integral multiples of 2π, which in effect means that the phase spectrum is zero.

Let us now consider one-sided filters. A *one-sided* or *causal* filter is defined as a filter with weights a_n that vanish for negative n. An important subclass of causal stable filters are the minimum-delay filters [1]. A strictly *minimum-delay filter* a_n is defined as a causal stable filter with z-transform

$$A(z) = \sum_{n=0}^{\infty} a_n z^n$$

analytic and without zeros on and within the unit circle, i.e., for $|z| \leqslant 1$.

As we might expect, a minimum-delay filter may be characterized by its phase spectrum. The unit circle $|z| = 1$ in the z-plane is mapped by the function $w = A(z)$ into a curve Γ in the w-plane. Since the equation for the unit circle is $z = e^{i\omega}$, we see that the curve Γ is $A(e^{i\omega})$, which we recognize as the frequency spectrum. The interior of the unit circle maps into the interior of the curve Γ. Since the z-transform $A(z)$ of a minimum-delay filter has no zeros on or inside the unit circle, the curve Γ never touches the origin $w = 0$ and its interior does not include the origin. We recall that the phase spectrum $\Theta(\omega)$ is defined as the argument of $A(e^{i\omega})$. Thus for a given ω, the value of $\Theta(\omega)$ is the angle of the vector from the origin $w = 0$ to the point $A(e^{i\omega})$ on the curve Γ. If we let ω go from $-\pi$ to π, then $z = e^{i\omega}$ will trace out the unit circle in the z-plane, and the vector $A(e^{i\omega})$ traces out the curve Γ in the w-plane. The angle of the vector $z = e^{i\omega}$ is the frequency ω, whereas the angle of the vector $A(e^{i\omega}) = |A(e^{i\omega})| \exp[i\Theta(\omega)]$ is the phase $\Theta(\omega)$. Since the curve Γ does not enclose the origin $w = 0$, the vector $A(e^{i\omega})$ cannot make a net rotation about the origin $w = 0$. In other words, as ω goes from $-\pi$ to π, any positive rotation of $A(e^{i\omega})$ must be balanced by the same amount of negative rotation. As a result the phase spectrum $\Theta(\omega)$ must return to its initial value. Thus, for a minimum-delay filter we have $\Theta(\pi) = \Theta(-\pi)$. However, for a non-minimum-delay filter, the curve Γ will enclose the origin $w = 0$. As ω goes from $-\pi$ to π, the vector $A(e^{i\omega})$ will make one complete rotation of the origin for each zero of $A(z)$ within the unit circle. Hence, a non-minimum-delay filter with N zeroes within the unit circle will have a net phase change of

$$\Theta(\pi) - \Theta(-\pi) = 2\pi N.$$

We can now characterize a minimum-delay filter by its phase spectrum: A causal stable filter is minimum-delay if and only if the net change in phase is zero over the Nyquist range, that is, if and only if

$$\Theta(\pi) - \Theta(-\pi) = 0.$$

For this reason minimum-delay filters are also called minimum-phase filters.

3. Positive Real Filters

Another important subclass of causal stable filters are the positive-real filters. A *positive-real filter* b_n is defined as a causal stable filter with a frequency spectrum $B(e^{i\omega})$ that has a positive-real part, that is,

$$\operatorname{Re} B(e^{i\omega}) > 0 \qquad \text{for } -\pi \leqslant \omega \leqslant \pi.$$

For filters with real weights b_n, the above condition becomes

$$\sum_{n=0}^{\infty} b_n \cos \omega n > 0.$$

Thus a causal stable filter with real weights is positive-real if and only if its cosine transform is positive.

Let us now define a symmetric filter c_n by

$$c_0 = b_0 ,$$

$$c_n = c_{-n} = 0.5\, b_n \qquad \text{for } n > 0.$$

Then we see that

$$\sum_{n=0}^{\infty} b_n \cos \omega n = \sum_{n=-\infty}^{\infty} c_n \cos \omega n$$

This equation states that

$$\text{Re } B(e^{i\omega}) = C(e^{i\omega}),$$

so if one is positive, the other will also be positive. Thus b_n is a positive-real filter if and only if c_n is a positive-definite filter.

Because Re $B(e^{i\omega}) > 0$ for a positive-real filter, it follows that its phase spectrum

$$\Theta(\omega) = \tan^{-1} \frac{\text{Im } B(e^{i\omega})}{\text{Re } B(e^{i\omega})}$$

must be bounded within the range $-\pi/2 < \Theta(\omega) < \pi/2$. That is, the curve Γ which represents the mapping $w = B(e^{i\omega})$ lies entirely within the right-half w-plane:

$$\text{Re } w = \text{Re } B(e^{i\omega}) > 0.$$

Because Γ does not touch or include the origin $w = 0$, we see that the filter b_n must be minimum-delay. Thus a positive-real filter is necessarily minimum-delay. The converse is not true, as a minimum-delay filter may or may not be positive-real.

We can summarize as follows:

(1) The phase spectrum of a symmetric filter can take on only values 0, $\pm\pi$, $\pm 2\pi$, $\pm 3\pi, \ldots$.

(2) The phase spectrum of a positive-definite filter can take on only the values 0, $\pm 2\pi$, $\pm 4\pi$ (which in effect is the value 0).

(3) The phase spectrum of a minimum-delay filter can take on any value as long as its value at π is the same as its value at $-\pi$.

(4) The phase spectrum of a positive-real filter can take on only values in the range $-\pi/2$ to $\pi/2$, and its value at π is the same as its value at $-\pi$.

4. Numerical Tests

In the previous sections we have given some of the characteristics of positive-definite filters, minimum-delay filters, and positive-real filters. Let us now deal with some numerical procedures for testing these properties. We confine ourselves to filters with a finite number of real weights. Such digital filters are called finite-length real-valued filters.

A finite-length filter $a_0, a_1, \dots, a_M$ is minimum-delay provided all the roots of the polynomial

$$A(z) = a_0 + a_1 z + \cdots + a_M z^M$$

lie outside the unit circle in the z-plane. As we have seen, the concepts of minimum-delay and minimum-phase are identical. Note that the minimum-phase property is a frequency-domain concept in that it pertains to the behavior of the phase of the complex quantity $A(z)$ for values of z on the unit circle. However, the equivalent root positioning requirement for minimum-delay is more directly associated with the time-domain values $a_0, a_1, \dots, a_M$ which determine these roots.

Similarly, the positive-real property is a frequency-domain concept. Its time-domain counterpart is less obvious, because the positive-real property bears no simple relation to the polynomial or its roots. However, the positive real property of a causal filter $b_0, b_1, \dots, b_M$ is equivalent to the positive-definite property of the corresponding symmetric (and thus noncausal) filter $c_{-M}, \dots, c_{-1}, c_0, c_1, \dots, c_M$ given by

$$c_0 = b_0,$$
$$c_n = c_{-n} = 0.5 b_n \qquad \text{for } n = 1, 2, \dots, M.$$

As we have seen, the positive-definite property of the symmetric filter is related to the Toeplitz form with kernel given by the filter weights.

The checking of the properties of positive-definite, minimum-delay, and positive-real can be made in the frequency domain. However, in digital computation we are limited to a finite set of spectral frequencies. For example in the use of the fast Fourier transform (FFT), we supplement the weights $a_0, a_1, \dots, a_M$ by zeros to give a signal $a_0, a_1, \dots, a_M, 0, 0, \dots, 0$ of length N, where N is a power of 2. The FFT program then computes the spectral values at the discrete set of frequencies $\omega_j = 2\pi j/N$, where $j = 1, 2, \dots, N-1$. Thus one must choose N sufficiently large so that density of the discrete frequencies ω_j is great enough to exhibit all the essential features of the frequency spectrum. The choice of N is not an easy problem from a mathematical point of view, so in applications one would have to be guided by experience. For example, if one wants to compute a cosine transform to check the positive-real property, then it is important to chose N large enough so that no negative portion of the transform is missed because that portion lies between two adjacent discrete frequencies. Similarly, an erroneous phase spectrum can be obtained simply by not picking a dense enough set of discrete frequencies. In brief, the accuracy of frequency-domaine tests depends upon a dense enough distribution of discrete frequencies.

More precise ways of checking the properties of minimum-delay, positive-real, and positive-definite properties can be formulated in the time-domain. Let us now give an effective test of the minimum-delay condition. First, let us make some definitions. Let

$$A_M(z) = a_0 + a_1 z + a_1 z^2 + \cdots + a_M z^M$$

be a polynomial of degree M with real-valued coefficients. Then the *reverse polynomial* is defined as

$$A_M{}^R(z) = a_M + a_{M-1} z + \cdots + a_0 z^M.$$

We see that for $z \neq 0$ the equality

$$A_M{}^R(z) = z^M A_M(z^{-1})$$

holds. From this equation it follows that

$$| A_M{}^R(z) | = | A_M(z^{-1}) | \qquad \text{for } | z | = 1,$$

which gives

$$| A_M{}^R(e^{i\omega})| = | A_M(e^{-i\omega})| = | A_M(e^{i\omega})|.$$

In words, a causal filter and its reverse causal filter have spectra of the same magnitude.

Now we want to scale the polynomial $A_M(z)$ so that its leading coefficient is unity. Given the polynomial $A_M(z)$ of degree $M \geqslant 1$ with leading coefficient unity, the *reflection coefficient* ρ_M is defined as the coefficient of z^M; that is, for the polynomial $1 + a_1 z + a_2 z^2 + \cdots + a_M z^M$, the reflection coefficient is defined as

$$\rho_M = a_M .$$

The (two-way) *transmission coefficient* is then defined as

$$\tau_M = 1 - a_M{}^2.$$

The polynomial A_{M-1} of degree $M - 1$ can then be defined as

$$A_{M-1}(z) = \tau_M^{-1}[A_M(z) - \rho_M A_M{}^R(z)].$$

We note that the leading coefficient of $A_{M-1}(z)$ is also unity. Next, we define the reflection coefficient ρ_{M-1} as the coefficient of z^{M-1} in $A_{M-1}(z)$. Likewise, the transmission coefficient is $\tau_{M-1} = 1 - \rho_{M-1}^2$. Continuing, we obtain,

$$A_{M-2}(z) = \tau_{M-1}^{-1}[A_{M-1}(z) - \rho_{M-1} A_{M-1}^R(z)].$$

Thus this algorithm allows us to compute the sequences $A_k(z)$, ρ_k, τ_k for $k = M$, $M - 1, \ldots, 2, 1$. Finally, we obtain $A_0(z) = 1$. We assume that no reflection coefficient has magnitude one, so no transmission coefficient is zero.

With this background, we give the following *theorem*:

REFLECTION COEFFICIENT THEOREM. *Let the polynomial*

$$A(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_M z^M$$

be the z-transform of a finite-length real-valued digital filter with leading coefficient $a_0 = 1$. *Define the polynomial* $A_N(z)$ *as the given polynomial* $A(z)$, *and carry out the above process to yield the reflection coefficients* ρ_M, $\rho_{M-1}, \ldots, \rho_2$, ρ_1. *The filter is minimum-delay if and only if each reflection coefficient has magnitude less than one.*

First let us prove the necessary statement in the theorem. Making use of an inductive proof, we assume that $A_k(z)$ is minimum-delay, which means that $A_k(z)$ has no zeros in $|z| \leqslant 1$. Because $a_0 = 1$, the coefficient ρ_k of z^k in $A_k(z)$ is equal to the product of the negative reciprocals of the zeros. Because all the zeros have magnitude greater than one, it follows that ρ_k must have magnitude less than one; i.e., $|\rho_k| < 1$. Thus, $\tau_k > 0$. Let us make use of Rouché's theorem [2] with the unit circle as the curve. We see that the right-hand side of the equation

$$A_{k-1}(z) = \tau_k^{-1} A_k(z) - \tau_k^{-1} \rho_k A_k{}^R(z)$$

the polynomials $A_k(z)$ and $A_k{}^R(z)$ each have the same magnitude on the unit circle. Because $|\rho_k| < 1$ it follows that the term $\tau_k^{-1} A_k(z)$ plays the role of the function with the "big" magnitude on the unit circle, whereas $\tau_k^{-1} \rho_k A_k{}^R(z)$ plays the role of the function with the small magnitude on the unit circle. Thus by Rouché's theorem the sum of these two functions, namely $A_{k-1}(z)$, has no zeros on the circumference of the unit circle, and has as many zeros inside as the "big" function, namely none. That is, $A_{k-1}(z)$ has no zeros in $|z| \leqslant 1$ and so is minimum-delay. Since, by hypothesis, $A(z)$ is minimum-delay, it follows by induction that: (i) Each member of the sequence $A_M(z) = A(z), A_{M-1}(z), \ldots, A_2(z), A_1(z)$ is minimum-delay, (ii) each reflection coefficient ρ_M, $\rho_{M-1}, \ldots, \rho_2$, ρ_1 is less than unity in magnitude, and (iii) each transmission coefficient τ_M, $\tau_{M-1}, \ldots, \tau_2$, τ_1 is positive. Q.E.D.

Next, let us prove the sufficient statement in the theorem. By hypothesis, each of the reflection coefficients ρ_1, $\rho_2, \ldots, \rho_M$ has magnitude less than one. Hence, it follows, as in the necessary part of the proof, that $A_{k-1}(z)$ has the same number of zeros as $A_k(e)$, where $k = 1, 2, \ldots, M$. Since $A_0(z) = 1$ has no zeros, it follows by mathematical induction that $A_1(z), A_2(z), \ldots, A_{M-1}(z), A_M(z) = A(z)$ each have no zeros in $|z| \leqslant 1$, and thus each is minimum-delay. Q.E.D.

The Reflection Coefficient Theorem gives an effective numerical algorithm for testing whether or not a digital filter is minimum-delay. Briefly, the algorithm tells us how to take the filter weights $a_0 = 1$, a_1, $a_2, \ldots, a_M$ and convert them into reflection coefficients ρ_1, ρ_2, ρ_3,..., ρ_M. The theorem states that the filter is minimum-delay if and only if $|\rho_k| < 1$ for $k = 1, 2,..., M$. It can be shown that this procedure is numerically equivalent to the Schur–Cohn test for determining whether a given polynomial is free of zeros in the closed unit disc [3]. In the Schur–Cohn test, it is not required that $a_0 = 1$. However, the restriction $a_0 = 1$ represents only a change of scale factor, and does not affect the generality of our theorem.

Let us now give a test for determining whether a finite-length symmetric filter is positive-real. We have the following theorem:

POSITIVE-DEFINITE THEOREM. *A necessary and sufficient condition that the symmetric filter* c_0, $c_1 = c_{-1}$, $c_2 = c_{-2}$,..., $c_M = c_{-M}$ *(with* $c_0 > 0$*) be positive-definite is that*

$$c_0 + 2c_1 + 2c_2 + \cdots + 2c_M > 0$$

and that

$$P(u) = p_0 + p_1 u + p_2 u^2 + \cdots + p_M u^M$$

obtained from

$$C(z) = c_0 + c_1(z + z^{-1}) + c_2(z^2 + z^{-2}) + \cdots + c_M(z^M + z^{-M})$$

by the substitution $u = 0.5(z + z^{-1})$ *has no real zero in the interval* $-1 \leqslant u \leqslant 1$.

The proof of this theorem follows from the following observation. A real value of u such that $|u| \leqslant 1$ corresponds to a value of z such that $|z| = 1$, as seen by the equation

$$u = 0.5(z + z^{-1}) = 0.5(e^{i\omega} + e^{-i\omega}) = \cos \omega.$$

That is, there is a one-to-one correspondence between each value of u on the real interval $-1 \leqslant u \leqslant 1$ and each value of z on the semicircle $z = e^{i\omega}$, where $0 \leqslant \omega \leqslant \pi$. If the symmetric filter is positive-definite, then its frequency spectrum

$$C(e^{i\omega}) = c_0 + c_1(e^{i\omega} + e^{-i\omega}) + \cdots + c_M(e^{i\omega M} + e^{-i\omega M})$$

is positive, which means $C(z)$ has no zero on the unit circle $z = e^{i\omega}$, or equivalently that $P(u)$ has no real zero u in the interval $-1 \leqslant u \leqslant 1$. In particular, $C(1) > 0$, which says that the sum of the filter coefficients is positive.

On the other hand, if $P(u)$ has no real zero in the interval $-1 \leqslant u \leqslant 1$, then $z = e^{i\omega}$ is not a zero of $C(z)$. Thus the frequency spectrum $C(e^{i\omega})$ does not vanish for any real ω. Because $c_0 + 2c_1 + \cdots + 2c_M = C(1)$ is positive, it follows that $C(e^{i\omega})$ is positive for all ω. We note that this Positive-Definite Theorem corresponds to a theorem of Wold [4] on the autocorrelation of a process of moving averages.

The Positive-Definite Theorem can be used to test whether a symmetric filter has zero-phase shift. In using the test we note that there is a simple recursive formula that can be used to obtain the polynomial $P(u)$ from the weights c_n. This is the Chebyshev recursion,

$$T_{n+1}(u) = 2uT_n(u) + T_{n-1}(u),$$

where $T_0(u) = 1$ and $T_1(u) = u$. Then the polynomial $P(u)$ is

$$P(u) = c_0 + 2c_1T_1(u) + 2c_2T_2(u) + \cdots + 2c_MT_M(u).$$

Finally we note that we can test the filter $b_0, b_1, b_2, \ldots, b_M$ for positive-real by testing the symmetric filter c_n defined by $c_0 = b_0$, $c_{-1} = c_1 = b_1/2$, $c_{-2} = c_2 = b_2/2, \ldots$, $c_{-M} = c_M = b_M/2$ for positive-definite.

References

1. E. A. Robinson and S. Treitel, "The Robinson–Treitel Eeader," 3rd ed., Seismograph Service Corporation, Tulsa, Okl., 1973.
2. M. Marden, "Geometry of Polynomials," Amer. Math. Soc., Providence, R.I., 1966.
3. D. Loewenthal, "Topics in Numerical Analysis," Vol. III, pp. 237–255, Academic Press, London, 1977.
4. H. Wold, "A Study in the Analysis of Stationary Time Series," 2nd ed., Almqvist & Wiksells, Stockholm, 1954.